Auction Theory Reading Notes

Eric Hsienchen Chu*

Spring, 2024

(⊛) Suggested readings: Prof. Marzena Rostek primarily uses Mas-Colell et al. (1995), Gibbons (2005), and Jehle and Reny (2010). As a side material, this reading notes is based on Krishna (2010), Chapter 1–4, from (TA) Rodrigo Yanez Naudon's suggestion and his Discussion handouts.

Contents

*Department of Economics, University of Wisconsin-Madison. hchu38@wisc.edu. This is reading notes for the second half of ECON713: Microeconomic Theory II. Instructor: Prof. Marzena Rostek. TA: Rodrigo Yanez Naudon.

1 Introduction

Overview. Here is a big picture of some common auction forms:

- **Open-bid Auction**
  - Descending/Dutch Auction: Price starts high. The winning bids pays at the price when the first bidder bids.
  - Ascending/English Auction: Price starts low. The winning bids pays the value when second-last bidder drops out.
- **Sealed-bid Auction**
  - First-price Auction (FPA): highest bid wins and pays the exact amount. (★)
  - Second-price Auction (SPA): highest bid wins but pays the second-highest bid.
  - All-pay Auction (APA): highest bid wins, but everyone pays for own bid.

Spoiler Alert 1.1. Descending/Dutch Auction is strategically equivalent to First-price Auction (FPA). And, Ascending/English Auction is *weakly* (strategically) equivalent to Second-price Auction (SPA) if with Independent Private Value (IPV).

2 Private Value Auctions

2.1 The Symmetric Model

Model 2.1 (Symmetric Model)**.** We make standard assumptions:

- Goods: single object
- Players: $\mathcal{I} = \{1, \cdots, I\}$ potential *risk neutral* bidders
- Valuation: bidder i assigns value of v_i to the object, where $v_i \in [0, V]$ and $v_i \overset{iid}{\sim} F$ for increasing CDF F.
- Common knowledge: the distribution F (of v) is common knowledge

Remark. Why do we care about symmetric equilibrium (BNEs)? It's an equilibrium in which all bidders follow the same bidding strategy $\mathbf{b}(\mathbf{v})$.

Remark. The "risk neutral" assumption will be useful when discussing Revenue Equivalence Theorem (**RET**).

2.2 Second-price Auction (SPA)

Recall that SPA is equivalent to Ascending/English Auctions with IPV.

Model 2.2 (SPA)**.** The payoffs of bidder i who bids $b_i(v_i)$ in SPA is:

$$u_i(v_i, b_i, b_{-i}) = \begin{cases} v_i - \max_{j \neq i} b_j, & \text{if } b_i > \max_{j \neq i} b_j =: b_{(2)} \\ 0, & \text{if } b_i < \max_{j \neq i} b_j =: b_{(2)} \end{cases} \tag{2.1}$$

Then, bidder i maximizes EU: $\max_{b_i} \mathbb{E}[u_i(v_i, b_i, b_{-i})|v_i, b_i] = (v_i - b_{(2)})\mathbb{P}(b_i > b_j,\ \forall j \neq i)$

Proposition 2.1. In SPA (w/IPV), the weakly dominant strategy is to bid $\mathbf{b}^{\text{SPA}}(\mathbf{v}) = \mathbf{v}$.

Proof. WLOG, let $b_i(v_i) = v_i$ be the winning bid. Suppose i bids $v_i^{'}$ s.t. $v_i^{'} < v_i$ (underbid). **IF** $v_i > v_i^{'} \geq b_{(2)}$, then i wins with payoff equals to $v_i - b_{(2)}$. **IF** $b_{(2)} > v_i > v_i^{'}$, then i loses the auction with payoff 0. But, **IF** $v_i > b_{(2)} > v_i^{'}$, then i loses the auction whereas bidding v_i yields positive payoffs. Thus, bidding *anything less than* v_i is weakly dominated by bidding exactly v_i. By symmetry, bidding *anything higher than* v_i is weakly dominated by bidding exactly v_i. Therefore, the unique symmetric BNE here is to bid own valuation v_i. □

Example 2.1 (Expected Payment in SPA)**.** Bidder i bids $\mathbf{b}^{SPA}(v_i) = v_i$ **but** pays only the second price. Define $G(v) := F(v)^{I-1}$, then expected payment by a bidder with value v_i is:

$$\mathbb{E}[\text{Payment}_i^{SPA}|v = v_i] = \mathbb{P}(b_i > b_j,\ \forall j \neq i) \times \mathbb{E}[b_{(2)}|b_{(1)} = v_i] \tag{2.2}$$

$$= \cancel{G(v_i)} \int_0^{v_i} y \frac{G'(y)}{\cancel{G(v_i)}} dy = \int_0^{v_i} yG'(y)dy \ (\bigstar) \tag{2.3}$$

Exercise 2.1 (Spring24 PS1 Q4(a) ($\bigstar$))**.** Consider an auction of a single object with I risk-neutral bidders with IPV for the object $v_i \overset{iid}{\sim} U[0, V]$.

(a) What would be the expected payment of a bidder if the auction format was a **Second-price (sealed-bid) Auction**.

Solution (a)**.** Consider $G'(v_i) = (I-1)F(v_i)^{I-2} = (I-1)(\frac{v_i}{V})^{I-2}$ and use Equation (2.3):

$$\mathbb{E}[\text{Payment}_i^{SPA}|v = v_i] = \cancel{G(v_i)} \int_0^{v_i} y \frac{G'(y)}{\cancel{G(v_i)}} dy = \int_0^{v_i} yG'(y)dy \tag{2.4}$$

$$= \int_0^{v_i} y(I-1)\frac{y^{I-2}}{V^{I-1}} dy = \left(\frac{I-1}{I}\right)\left(\frac{v_i^I}{V^{I-1}}\right) \tag{2.5}$$

(*Alternatively, we can solve this by* $\mathbf{b}^{SPA}(v_i) = \mathbf{b}^{FPA}(v_i)G(v_i) = \mathbb{E}[b_{(2)}|b_{(1)} = v_i]G(v_i)$.)

2.3 First-price Auction (FPA)

Model 2.3 (FPA)**.** The payoffs of bidder i who bids $b_i(v_i)$ in FPA is:

$$u_i(v_i, b_i, b_{-i}) = \begin{cases} v_i - b_i, & \text{if } b_i > b_{-i} \\ \frac{1}{2}(v_i - b_i), & \text{if } b_i = b_{-i} \\ 0, & \text{if } b_i < b_{-i} \end{cases} \tag{2.6}$$

Assuming atomless distribution $v \overset{iid}{\sim} F[0,1]$ Then, bidder i maximizes EU:

$$\begin{aligned}
\max_{b_i} \mathbb{E}[u_i(v_i, b_i, b_{-i})|v_i, b_i] &= (v_i - b_i)\mathbb{P}(b_i > b_j,\ \forall j \neq i) && (2.7)\\
&= (v_i - b_i)F(v_i)^{I-1} && (2.8)\\
&= (v_i - b_i)F(b^{-1}(b(v_i)))^{I-1} && (2.9)\\
&= (v_i - b_i)G(b^{-1}(b(v_i))) && (2.10)\\
\longrightarrow FOC\ [b_i]:\ 0 &= -G(v_i) + (v_i - b_i)G'(v_i)\overbrace{[b^{-1}(b(v_i))]'}^{=\frac{1}{b'(v_i)}} && (2.11)\\
0 &= -G(v_i)b'(v_i) + (v_i - b_i)G'(v_i) && (2.12)\\
G(v_i)b'(v_i) + b(v_i)G'(v_i) &= v_iG'(v_i) && (2.13)\\
\int_0^{v_i} \frac{\partial[G(y)b(y)]}{\partial y}dy &= \int_0^{v_i} yG'(y)dy && (2.14)\\
\implies \mathbf{b}^{FPA}(v_i) &= \frac{1}{G(v_i)}\int_0^{v_i} yG'(y)dy\ (\bigstar) && (2.15)\\
&= \frac{1}{G(v_i)}\left[\Big[yG(y)\Big]_0^{v_i} - \int_0^{v_i} G(y)dy\right] && (2.16)\\
&= v_i - \int_0^{v_i} \frac{G(y)}{G(v_i)}dy\ (\bigstar) && (2.17)
\end{aligned}$$

Remark (Intuition)**.** From Equation (2.15), we learn that $\mathbf{b}^{FPA}(v_i) = \frac{1}{G(v_i)}\int_0^{v_i} yG'(y)dy$. Specifically, as hinted in SPA, the symmetric BNE in FPA is to **underbid**:

$$\begin{aligned}
\mathbf{b}^{FPA}(v_i) = \frac{1}{G(v_i)}\int_0^{v_i} yG'(y)dy &= \mathbb{E}[b_{(2)}|b_{(1)} = v_i] && (2.18)\\
&= \mathbb{E}[\max_{j\neq i} v_j|v_i > v_j,\ \forall j \neq i], && (2.19)
\end{aligned}$$

where $b_{(2)} := \max_{j\neq i} v_j$ is the highest of $I-1$ values (i.e., *first-order statistics*). Essentially, no bidder would bid own valuation since payoff is 0. The bidder thus faces a simple *trade-off*: an increase in the bid increases the probability of winning but reduces the payoffs.

Remark. In BNE, the bidder with the highest valuation wins the auction since $\mathbf{b}(\mathbf{v})$ is strictly increasing and continuous function (monotonic).

Remark (Bid Shading)**.** The bid is naturally less than v_i since $\frac{G(y)}{G(v_i)} = \left[\frac{F(y)}{F(v_i)}\right]^{I-1}$, the degree of "shading" (the amount by which the bid $\mathbf{b}(v_i)$ is less than v_i) depends on I. For a fixed distribution F, the bid shading approaches to 0 as I increases.

Example 2.2 (FPA with Uniform)**.** Suppose there are I risk neutral bidders with value $v \overset{iid}{\sim} U[0,1]$. Define $G(v) := F(v)^{I-1} = v^{I-1}$ (by *Uniform*), then by Equation (2.15)

$$\begin{aligned}
\mathbf{b}^{FPA}(v_i) &= \frac{1}{G(v_i)} \int_0^{v_i} yG'(y)dy && (2.20) \\
&= \frac{1}{v_i^{I-1}} \int_0^{v_i} y(I-1)y^{I-2}dy && (2.21) \\
&= \frac{1}{v_i^{I-1}} \left(\frac{I-1}{I} v_i^I\right) = \frac{I-1}{I} v_i && (2.22)
\end{aligned}$$

Example 2.3 (FPA with Exoponential)**.** Suppose there are 2 risk neutral bidders with value $v \overset{iid}{\sim} Exp(\lambda)$ on $[0,\infty)$ $(\lambda > 0)$. Define $G(v) := F(v)^{2-1} = F(v) = 1 - e^{-\lambda v}$ (by *Exponential*), then by Equation (2.17)

$$\begin{aligned}
\mathbf{b}^{FPA}(v_i) &= v_i - \int_0^{v_i} \frac{G(y)}{G(v_i)} dy && (2.23) \\
&= v_i - \frac{1}{(1-e^{-\lambda v_i})} \int_0^{v_i} 1 - e^{-\lambda y} dy && (2.24) \\
&= v_i - \frac{1}{(1-e^{-\lambda v_i})} \left(v_i + \frac{1}{\lambda} e^{-\lambda v_i} - \frac{1}{\lambda}\right) && (2.25) \\
&= \frac{1}{\lambda} - \frac{v_i e^{-\lambda v_i}}{1 - e^{-\lambda v_i}} && (2.26)
\end{aligned}$$

In a special case where $\lambda = 2$, we notice $\mathbf{b}^{FPA}(v_i) < \frac{1}{2}$, i.e., bidders with high values are still only willing to bid a very small amount.

2.4 Revenue Comparison

Motivation. We have derived the (symmetric) optimal bidding strategy in FPA & SPA. We now want to compare the expected revenues from the two auction formats.

Fact 2.1 (Payment/Revenue Equivalence: FPA & SPA)**.** We notice that the *expected payment by a bidder with valuation* v_i is:

$$\begin{aligned}
\mathbb{E}[\text{Payment}_i^{FPA}|v=v_i] &= \underbrace{\mathbf{b}^{FPA}(v_i)}_{\text{amount paid}} \times \underbrace{\mathbb{P}(b_i > b_j,\ \forall j \neq i)}_{\text{prob. of winning}} && (2.27)\\
&= \mathbb{E}[b_{(2)}|b_{(1)} = v_i] \times G(v_i) && (2.28)\\
&= \cancel{G(v_i)} \int_0^{v_i} y \frac{G'(y)}{\cancel{G(v_i)}} dy && (2.29)\\
&= \int_0^{v_i} yG'(y)dy = \mathbb{E}[\text{Payment}_i^{SPA}|v=v_i] \ (\bigstar) && (2.30)
\end{aligned}$$

Further suppose $v \overset{iid}{\sim} F[0,1]$ with density f. Since ① the expected payment by bidder with v_i is the same between FPA & SPA and that ② *expected revenue* is the sum of the "*ex ante* expected payment", we observe the **revenue equivalence**:

$$\begin{aligned}
ER^{FPA} &= I \times \underbrace{\mathbb{E}[\text{Payment}_i^{FPA}]}_{\text{ex ante Exp. Payment}} && (2.31)\\
&= I \times \int_0^1 \mathbb{E}[\text{Payment}_i^{FPA}|v=v_i] \cdot \underbrace{f(v)}_{\text{density}} dv && (2.32)\\
&= I \times \int_0^1 \mathbb{E}[\text{Payment}_i^{SPA}|v=v_i] \cdot f(v)dv \longleftarrow \text{by Equation (2.30)} && (2.33)\\
&= I \times \mathbb{E}[\text{Payment}_i^{SPA}] && (2.34)\\
&= ER^{SPA} && (2.35)
\end{aligned}$$

Proposition 2.2. With *iid* private values, the *Expected Revenue* in a FPA is the same as the *Expected Revenue* in a SPA.

Remark. While the revenue may be greater in one auction or another depending on the realized values, we have argued that *on average* the revenue will be the same in FPA & SPA.

Remark. We can actually extend such revenue equivalences to more general auctions, which we will introduce the Revenue Equivalence Theorem (**RET**) in Section (3.1).

2.5 Reserve Prices

Motivation. In many instances, sellers reserve the right to *not* sell the object if the price determined in the auction is lower than *reserve price* $r > 0$.

Model 2.4 (Reserve Price in SPA)**.** With a reserve price $r > 0$, only bidders with value $v_i \geq r$ will bid in the auction. No change to the weakly dominant strategy by bidding own valuation $\mathbf{b}^{SPA}(v_i) = v_i$. At the cutoff, bidder of value r will bid r. The expected payment by a bidder of value $v_i \geq r$ is given by:

$$m^{SPA}(v_i; v_i \geq r) = \underbrace{rG(r)}_{\text{baseline}} + \underbrace{\int_r^{v_i} yG'(y)dy}_{\text{for } v_i \geq r} \; (\bigstar) \tag{2.36}$$

Remark (Intuition)**.** The winner pays the reserve price r whenever the second-highest bid is below r, governed by $rG(r)$. The second part is from Equation (2.30) and modify the lower bound of integral.

Model 2.5 (Reserve Price in FPA)**.** Similarly, with a reserve price $r > 0$, only bidders with value $v_i \geq r$ will bid in the auction. At the cutoff, bidder of value r will bid r. Modifying Equation (2.14), we solve:

$$\int_r^{v_i} \frac{\partial[G(y)b(y)]}{\partial y} dy = \int_r^{v_i} yG'(y)dy \tag{2.37}$$

$$\implies G(v_i)b(v_i) - G(r)b(r) = \int_r^{v_i} yG'(y)dy \tag{2.38}$$

$$\implies \mathbf{b}^{FPA}(v_i) = \frac{1}{G(v_i)} \left[b(r)G(r) + \int_r^{v_i} yG'(y)dy \right] \tag{2.39}$$

$$= \frac{1}{G(v_i)} \left[rG(r) + \int_r^{v_i} yG'(y)dy \right] \tag{2.40}$$

Then, the expected payment by a bidder of value $v_i \geq r$ is given by:

$$m^{FPA}(v_i) = \mathbf{b}^{FPA}(v_i) \cdot \mathbb{P}(b_i > b_j, \ \forall j \neq i) \tag{2.41}$$

$$= \mathbf{b}^{FPA}(v_i) \cdot G(v_i) \tag{2.42}$$

$$= \; = \underbrace{rG(r)}_{\text{baseline}} + \underbrace{\int_r^{v_i} yG'(y)dy}_{\text{for } v_i \geq r} \; (\bigstar) \tag{2.43}$$

Remark. By Revenue Equivalence, Equation (2.36) equals Equation (2.43). Thus, with reserve price $r > 0$, the expected payments and expected revenue will all again be the **same**.

Exercise 2.2 (Spring24 TA Handout 8 Q4 Modified (★★★)). Suppose there are I bidders in a FPA. The valuation of the bidders v is private information drawn from $v \overset{iid}{\sim} F[0,1]$. Further suppose that the seller set a **reserve price** $r > 0$. What is the **revenue** of the auctioneer?

Solution. From Model (2.5), we obtain the expected payment by bidder of value v_i:

$$m^{FPA}(v_i; r) := \mathbb{E}[\text{Payment}_i^{FPA} | v = v_i, r] = rG(r) + \int_r^{v_i} yG'(y)dy \tag{2.44}$$

The *ex ante* expected payment conditional on r is then given by:

$$\begin{aligned}
\mathbb{E}[\text{Payment}_i^{FPA}] &= \int_r^1 m^{FPA}(v_i; r) \cdot f(v)dv && (2.45)\\
&= \int_r^1 \left(rG(r) + \int_r^{v_i} yG'(y)dy \right) f(v)dv && (2.46)\\
&= rG(r) \Big[F(v)\Big]_r^1 + \int_r^1 \underbrace{\left[\int_r^{v_i} yG'(y)dy\right]}_{\text{let } = h(v,r)\ (\bigstar)} f(v)d(v) && (2.47)\\
&= rG(r)\Big(1 - F(r)\Big) + \int_r^1 h(v,r)f(v)dv && (2.48)\\
&= rG(r)\Big(1 - F(r)\Big) + \Big[h(v,r)F(v)\Big]_r^1 - \int_r^1 F(v)\frac{d}{dv}\Big[yG'(y)\Big]_r^v dv && (2.49)\\
&= rG(r)\Big(1 - F(r)\Big) + \Big[h(v,r)F(v)\Big]_r^1 - \int_r^1 F(v)vG'(v)dv && (2.50)\\
&= rG(r)\Big(1 - F(r)\Big) + h(1,r)\cdot 1 - \underbrace{h(r,r)}_{=0} F(r) - \int_r^1 F(v)vG'(v)dv && (2.51)\\
&= rG(r)\Big(1 - F(r)\Big) + \int_r^1 vG'(v)dv \cdot 1 - \int_r^1 F(v)vG'(v)dv && (2.52)\\
&= rG(r)\Big(1 - F(r)\Big) + \int_r^1 \Big(1 - F(v)\Big) vG'(v)dv && (2.53)
\end{aligned}$$

Consider that the seller attaches a value $v_0 \in [0,1)$ if the object remains unsold. We notice that the seller will only set a reserve price r s.t. $r \geq v_0$.
Therefore, the overall Expected Revenue with reserve price $r \geq v_0$ is:

$$\Pi = I \times \mathbb{E}[\text{Payment}_i^{FPA}] + F(r)^I v_0 \tag{2.54}$$

In addition, we can solve optimal reserve price r^* and find that $r^* > v_0$:

$$FOC\ [r]:\ 0 = I[1 - F(r) - rf(r)]G(r) + I \cdot G(r)f(r)v_0\ (skip) \tag{2.55}$$

3 The Revenue Equivalence Principle

3.1 Main Results

Theorem 3.1 (Revenue Equivalence Theorem)**.** Consider $I \geq 2$ bidders. Suppose BNEs of any two auctions are such that:

① Bidders are *risk neutral*,

② $\{v_i\}_{i\in\mathcal{I}} \overset{iid}{\sim} F$,

③ $\forall$ valuation profile $(v_1, \cdots, v_I)$, the highest value bidder "i" has the same probability of winning the auction, and

④ the lowest value bidder has the same *ex post* payoff

Then, the Expected Revenue of the two auctions are the same.

Remark. The RET breaks when (i) risk-averse bidders, (ii) interdependent values, (iii) budget constraints, and (iv) collusion.

Exercise 3.1 (Spring24 PS1 Q4(b) (★))**.** Consider an auction of a single object with I risk-neutral bidders with IPV for the object $v_i \overset{iid}{\sim} U[0, V]$. The auctioneer sells the object through an *all pay auction*, defined as a simultaneous sealed-bid auction in which the higher bidder wins the object, but every bidder pays her submitted bid.

(b) Applying the **Revenue Equivalence Theorem**, solve for the bidding functions in a symmetric equilibrium in the *all-pay auction*.

Solution (b)**.** Consider APA:

$$u_i(v_i, b_i, b_{-i}) = \begin{cases} v_i - b_i, & \text{if } b_i > \max_{j\neq i} b_j =: b_{(2)} \\ -b_i, & \text{if } b_i < \max_{j\neq i} b_j =: b_{(2)} \end{cases} \tag{3.1}$$

$$\implies \max_{b_i} v_i G(b^{-1}(b(v_i))) - b_i \tag{3.2}$$

To invoke Theorem (3.1) (**RET**) between SPA and APA, we check the four conditions:

① Bidders are *risk neutral* (✓),

② $\{v_i\}_{i\in\mathcal{I}} \overset{iid}{\sim} U[0, 1]$: iid (✓),

③ the highest value bidder has the same winning prob $= F(v_i)^{I-1} = \left(\frac{v_i}{V}\right)^{I-1}$ (✓), and

④ the lowest value bidder has the same *ex post* payoff of 0 (SPA: 0; APA: $v_i - b_i = 0 - 0 = 0$) (✓)

$\implies$ We can apply **RET** and use Exercise (2.1) to find that:

$$\mathbf{b}^{APA}(v_i) = \underbrace{\mathbb{E}[\text{Payment}_i^{APA}|v = v_i] = \mathbb{E}[\text{Payment}_i^{SPA}|v = v_i]}_{\text{Revenue Equivalence Theorem}} = \left(\frac{I-1}{I}\right)\left(\frac{v_i^I}{V^{I-1}}\right) \quad (3.3)$$

> **Fact 3.1** (Shortcut for Expected Revenue)**.** Let $m^{\mathcal{A}}(v_i) := \mathbb{E}[\text{Payment}_i^{\mathcal{A}}|v = v_i]$ be the equilibrium expected payment in any auction $\mathcal{A}$ by a bidder with value v_i. Suppose $\mathbf{b}^{\mathcal{A}}(\mathbf{v})$ is such that $m^{\mathcal{A}}(0) = 0$, then:
>
> $$m^{\mathcal{A}}(v_i) = \mathbb{E}[b_{(2)}|b_{(1)} = v_i] \cdot G(v_i) \quad (3.4)$$
> $$= \int_0^{v_i} y\frac{G'(y)}{G(v_i)}dy \cdot G(v_i) = \int_0^{v_i} yG'(y)dy \ (\bigstar) \quad (3.5)$$

Remark. This result comes from $m^{\mathcal{A}}(v_i) = m^{\mathcal{A}}(0) + \int_0^{v_i} yG'(y)dy = \int_0^{v_i} yG'(y)dy$.

Example 3.1 (FPA & SPA)**.** In FPA & SPA (see Equation (2.30)), we note that:

$$m^{FPA}(v_i) = \underbrace{\mathbf{b}^{FPA}(v_i)}_{\text{amount paid}} \times \underbrace{\mathbb{P}(b_i > b_j,\ \forall j \neq i)}_{\text{prob. of paying}} = \int_0^{v_i} yG'(y)dy \quad (3.6)$$

$$m^{SPA}(v_i) = \underbrace{\mathbb{E}[b_{(2)}|b_{(1)} = v_i]}_{\text{amount paid}} \times \underbrace{\mathbb{P}(b_i > b_j,\ \forall j \neq i)}_{\text{prob. of paying}} = \int_0^{v_i} yG'(y)dy \quad (3.7)$$

Note: In either case, *Expected Revenue* is just the expectation of the second-highest value.

Example 3.2 (APA; special case)**.** Consider APA (see Exercise (3.1)) but now with $v \overset{iid}{\sim} U[0,1]$. Let's define $G(v) := F(v)^{I-1} = v^{I-1} \implies G'(v) = (I-1)v^{I-2}$. By Equation (3.5), we note that the expected payment by a bidder with value of v_i is:

$$m^{APA}(v_i) = \int_0^{v_i} yG'(y)dy = \int_0^{v_i} y(I-1)y^{I-2}dy = \frac{I-1}{I}v_i^I \quad (3.8)$$

Note: Bidder i bids $\mathbf{b}^{APA}(v_i) = \frac{I-1}{I}v_i^I = m^{APA}(v_i)$, which coincides with her expected payment. To see $\mathbf{b}^{APA}(v_i)$ we solve Equation (3.2).

> **Summary.** The Expected Payment by bidder of value v_i ($m^{\mathcal{A}}(v_i)$) is given by:
>
> - **FPA** : the bid $\times$ winning prob. $\implies \mathbf{b}^{FPA}(v_i)G(v_i) = \mathbb{E}[b_{(2)}|b_{(1)} = v_i]G(v_i)$
> - **SPA** : the EV of the $b_{(2)}$ cond. on being winning bid $\implies \mathbb{E}[b_{(2)}|b_{(1)} = v_i]G(v_i)$
> - **APA** : coincides with the bid itself. $\implies \mathbf{b}^{APA}(v_i)$

3.2 Applications of RET

(For this Chapter, I suggest checking out Exercises on TA Handouts and Past Exams.)

Definition 3.1 ("kth" Order Statistic)**.** Make n independent draws from a random variable with distribution F_Y. The distribution of the kth order statistic is given by:

$$F_{Y(k)}(v) = \sum_{j=k}^{N} \binom{N}{j} \left[F_Y(y)\right]^j \left[1 - F_Y(y)\right]^{N-j} \tag{3.9}$$

Exercise 3.2 (Spring24 TA Handout 9 Ex3)**.** Let $v \sim F[\underline{v}, \bar{v}]$. In a special case of "APA but pay the second-highest bid," we are interested in the distribution of **second-highest** value $\iff$ second-order statistics $\iff$ $N-1th$ highest value:

$$\begin{aligned} F^{II}(y) &= \sum_{j=N-1}^{N} \binom{N}{j} \left[F_Y(y)\right]^j \left[1 - F_Y(y)\right]^{N-j} && (3.10)\\ &= \binom{N}{N} \left[F_Y(y)\right]^N \left[1 - F_Y(y)\right]^0 + \binom{N}{N-1} \left[F_Y(y)\right]^{N-1} \left[1 - F_Y(y)\right]^1 && (3.11)\\ &= \left[F_Y(y)\right]^N + N \left[F_Y(y)\right]^{N-1} \left[1 - F_Y(y)\right]^1 && (3.12) \end{aligned}$$

For instance, in a 2-bidder Auction with such format ($N = 2$), $F^{II}(y)$ collapses to:

$$\begin{aligned} F^{II}(y) &= \left[F_Y(y)\right]^2 + 2\left[F_Y(y)\right]^1 \left[1 - F_Y(y)\right]^1 && (3.13)\\ \implies f^{II}(y) &= 2\left[1 - F(y)\right] f(y) && (3.14) \end{aligned}$$

The (conditional) expected payment and Expected Revenue are thus:

$$\begin{aligned} m^{II,SPA}(v_i) &= \mathbf{b}^{II,APA}(v_i) \cdot 2\left[1 - F(v_i)\right] f(v_i) && (3.15)\\ \implies ER^{II,SPA} &= 2 \cdot \int_{\underline{v}}^{\bar{v}} \mathbf{b}^{II,APA}(y) \cdot 2\left[1 - F(y)\right] f(y)dy && (3.16) \end{aligned}$$

References

Gibbons, R. (2005). A primer in game theory. Pearson.

Jehle, G. A., & Reny, P. J. (2010). Advanced microeconomic theory. Addison-Wesley.

Krishna, V. (2010). Auction theory. Academic Press. https://doi.org/10.1016/C2009-0-22474-3

Mas-Colell, A., Whinston, M. D., & Green, J. R. (1995). Microeconomic theory. Oxford University Press.